

Averaging of an autoresonance model

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Abstract

Autoresonance is a phase locking phenomenon occurring in nonlinear oscillatory system, which is forced by oscillating perturbation. Many physical applications of the autoresonance are known in nonlinear physics. The essence of the phenomenon is that the nonlinear oscillator selfadjusts to the varying external conditions so that it remains in resonance with the driver for a long time. This long time resonance leads to a strong increase in the response amplitude under weak driving perturbation. An analytic treatment of a simple mathematical model is done here by means of asymptotic analysis using a small driving parameter. The main result is finding threshold for entering the autoresonance.

1 Introduction

A hamiltonian system of two differential equations with a small parameter ε

$$u' - H_v(u, v) = \varepsilon f(\tau) \cos(\varphi), \quad v' + H_u(u, v) = \varepsilon g(\tau) \cos(\varphi). \quad (1)$$

is considered. Here the right hand side represents a prescribed small external force which is fast oscillating: $\varphi = \Phi(\tau)/\varepsilon$. The $f, g, \Phi(\tau)$ are smooth functions of the slow time $\tau = \varepsilon t$. It is supposed that the unperturbed system (as $\varepsilon = 0$) has the stable equilibrium point $(0, 0)$ of center type in general position, which is taken as the initial point of the perturbed solution

$$u, v(t; \varepsilon)|_{t=0} = 0. \quad (2)$$

In this note we give an asymptotic solution of the problem (1),(2) as $\varepsilon \rightarrow 0$, which is valid for large time $t = \mathcal{O}(\varepsilon^{-1})$. The main goal is to find conditions

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under which the system's energy $H(u, v) + \varepsilon[vf - ug] \cos(\varphi)$ grows up to the order of unity while the driver is being small: $\varepsilon f, \varepsilon g = o(1)$, $\varepsilon \rightarrow 0$. Such type phenomenon is referred to as autoresonance and one was studied by different authors [1-10].

Our construction is based on general two-parametric periodic solution of the unperturbed system: $u_0, v_0(t + t_0, E)$, $(t_0 \in \mathbb{R}, E \in (0, e_0) \subset \mathbb{R})$ which exists in a neighborhood of the equilibrium point $(0, 0)$. We set $E = 0$ in the equilibrium solution $u_0, v_0(t, 0) \equiv 0$. The parameter E is a first integral of the unperturbed system, hence one can be identified as unperturbed energy $E = H(u, v)$.

We consider the general case, when both the period $T = T(E)$ and the frequency $\omega = 2\pi/T(E)$ depend on the energy E so that $T'(E) \neq 0$. This dependence is a speciality of nonlinear system. It provides entering in autoresonance under different driver frequency.

2 Ansatz

An averaging method elaborated by Kuzmak [11] is here applied to asymptotic analysis of the solution. Sometimes this asymptotic approach is referred to as adiabatic approximation. The basic idea is that the leading order term of the asymptotic solution is taken as unperturbed solution

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0(\sigma, E) \\ v_0(\sigma, E) \end{pmatrix} + \varepsilon \begin{pmatrix} u_1(\sigma, \tau) \\ v_1(\sigma, \tau) \end{pmatrix} + \mathcal{O}(\varepsilon^2), \quad (\tau = \varepsilon t) \quad (3)$$

under appropriate slow deformation of both the phase $\sigma = \varepsilon^{-1}\Psi(\tau; \varepsilon)/\omega(E)$ and energy $E = E(\tau, \varepsilon)$. In the leading order term the problem is to find two functions $\Psi, E(\tau; \varepsilon)$ depending on slow time $\tau = \varepsilon t$. The main result is asymptotics of these functions.

$$\Psi(\tau; \varepsilon) = \Psi_0(\tau) + \varepsilon \Psi_1(\tau) + \varepsilon^2 \Psi_2(\tau) + \mathcal{O}(\varepsilon^3), \quad E(\tau, \varepsilon) = E_0(\tau) + \varepsilon E_1(\tau) + \mathcal{O}(\varepsilon^2) \quad (4)$$

The initial energy is zero $E(0, \varepsilon) = 0$. If we find that $E_0(\tau) = \mathcal{O}(1)$ as $\tau = \mathcal{O}(1)$, it means that the energy of the system goes to order of unity for large times $t = \mathcal{O}(\varepsilon^{-1})$.

3 The slow deformation equation

The property of T - periodic with respect to fast variable σ identifies a class of functions $u, v(\sigma, \tau)$ which are used for asymptotic solution. This secular condition, as applied to the first order correction, leads to the equations for the desired slow varying functions. In the leading order term we obtain

$$\Psi'_0 = \omega(E_0), \quad E'_0 = \frac{1}{T} \int_0^T [\partial_\sigma \mathbf{u}_0(\sigma, E_0), \mathbf{F}(\varphi; \tau)] d\sigma \quad (5)$$

Here the integrand is skew-symmetric product of the $\partial_\sigma \mathbf{u}_0 = (u'_0(\sigma, E), v'_0(\sigma, E))$ by the perturbation, which is $\mathbf{F} = (f, g) \cos(\varphi)$ in the case of (1). As usually in this way the relations (5) are looking as differential equations. But in the case under consideration we can convert theirs into algebraic equations. To this end one have to take into account that there is the fast variable $\varphi = \Phi(\tau)/\varepsilon$ with prescribed function $\Phi(\tau)$ in the original problem (1). It is not to hard to guess that in resonance case the desired fast variable σ must to be related with the given φ . Such type relation we refer to as

Resonance requirement. *Difference between driver frequency and free frequency is small*

$$\Psi' - \Phi' = \mathcal{O}(\varepsilon) \quad (6)$$

So we try to find asymptotic solution in the form (3),(4) under additional relation (6). This approach may be realized only under some restrictions which are just autoresonance conditions.

As the leading order term of the phase $\Psi_0(\tau)$ is the given function $\Phi_0(\tau)$ hence the first equation in (5) is reading as an algebraic equation

$$\omega(E_0) = \Phi'(\tau) \quad (7)$$

for the energy $E_0(\tau)$. Next, the second equation in (5) is reading as an algebraic equation for the first order correction of the phase function $\Psi_1(\tau)$. In order to see it we have to take into account the resonance relation (6) as follows $\varphi = \omega(E)\sigma - (\Psi - \Phi)/\varepsilon = \omega(E)\sigma - \Psi_1 + \mathcal{O}(\varepsilon)$.

Analysis of higher order corrections (u_n, v_n) , $n \geq 2$ leads to linear algebraic equations for the corrections of both the energy and the phase function in the expansion (4).

4 Results

It is clear that in general case the nonlinear algebraic equations have no solutions. Requirements, under which they are solved, yield conditions of entering the autoresonance.

Theorem 1 *Let both $\omega'(E) \neq 0$ and the driver frequency coincide with the free frequency in the initial moment $\Phi'(0) = \omega(0)$. Then the equation (7) has unique growing solution $E_0 = E_0(\tau)$ for all $\tau \in [0, \tau_0]$ if and only if directions of the frequency variations coincide $\text{sgn } \Phi''(\tau) = \text{sgn } \omega'(E)$.*

Now the second equation in (5) is reduced to the algebraic equation for the Ψ_1

$$-\frac{\Phi'(\tau)}{2\pi} \int_0^{2\pi} [\mathbf{u}_0(T(E_0)s/2\pi, E_0), \partial_s \mathbf{F}(s - \Psi_1; \tau)] ds = \Phi''(\tau)/\omega'(E_0) \quad (8)$$

The equation can be solved only under specific conditions on the driver data $\mathbf{F}(s, \tau)$, $\Phi(\tau)$. In particular, one can see from (8) that the property: $\Phi''(0) = 0$ is need, because the unperturbed solution tends to zero: $\mathbf{u}_0(T(E_0)s/2\pi, E_0) \rightarrow 0$

as energy tends to zero $E_0(\tau) \rightarrow 0$, $\tau \rightarrow 0$. A crucial condition is imposed on the amplitude. In order to clear situation we consider a specific driver as given in (1). In that case the (8) is reduced to the trigonometric equation as follows

$$\Phi'(\tau)[a(\tau) \cos(\Psi_1) - b(\tau) \sin(\Psi_1)] = \Phi''(\tau)/\omega'(E_0(\tau)) \quad (9)$$

Here a, b are Fourier coefficients depending on the unperturbed solution

$$a = \frac{1}{2\pi} \int_0^{2\pi} [g(\tau)u_0 - f(\tau)v_0] \sin(s) ds, \quad b = \frac{1}{2\pi} \int_0^{2\pi} [g(\tau)u_0 - f(\tau)v_0] \cos(s) ds$$

Thus the a, b are known as far as the unperturbed solution is known: $(u_0, v_0)(\sigma, E)$ under $\sigma = T(E)s/2\pi$, $E = E_0(\tau)$. The following assertion is easily proved.

Theorem 2 *Let both the the right hand side in the original problem be given as in (1) and conditions of theorem 1 hold. Then equation (8) is solved if and only if the inequality*

$$a^2(\tau) + b^2(\tau) \geq [\Phi''(\tau)/\omega\omega'(E_0(\tau))]^2 \quad (10)$$

holds.

Condition (10) may be simplified. To this end an asymptotics of the unperturbed solution as $E \rightarrow 0$ can be used. In this way the following assertion is obtained:

Corollar 2.1 *Let both $\Phi''(0) = 0$ and the strong inequality*

$$\omega^2(0)[f^2(0) + g^2(0)] > 4|\Phi'''(0)/\omega'(0)|$$

hold in the initial moment. Then the equation (9) has a solution $\Psi_1 = \Psi_1(\tau)$ on some finite interval $\tau \in [0, \tau_0]$.

5 Conclusion

Requirements of the theorems 1,2 may be considered as conditions under which the system enters autoresonance, so that the energy of the system grows up to order of unity for large times $t = \tau/\varepsilon = \mathcal{O}(\varepsilon^{-1})$. In particular inequality (10) may be interpreted as threshold for entering the autoresonance.

6 Restrictions

For the first we have to remark that the formulas (3),(4) give only a formal asymptotic solution. The problem of justification of the asymptotics remains open. Moreover there remains an ambiguity in the the phase shift of the leading order term because of two roots of the equation (9).

For the second we have to note that the formal asymptotic solution as given above is valid only for large times and one is not suitable on initial stage.

In order to see this nuisance one have to analyze higher order terms in the expansion (4). They are obtained from linear algebraic equations and in this way any additional requirements on the data do not arise. However higher order corrections have singularities as $\tau \rightarrow 0$, which became stronger as the number of the correction increases. For example, $E_1(\tau) = \mathcal{O}(\tau^{-1})$, $\Psi_2(\tau) = \mathcal{O}(\tau^{-3})$. Such type singularities indicate that the ansatz (3),(4) is not suitable on the initial stage, when $\tau = \varepsilon t$ is small. Formulas (3),(4) give an asymptotic solution only for large times $\varepsilon^{-2/3} \ll t \leq \mathcal{O}(\varepsilon^{-1})$.

Asymptotic solution on the initial stage must be taken in another form in which the slow time $\theta = \varepsilon^{2/3}t$ is used [12]. Such structure of the asymptotics is similar to boundary layer. Solution which is valid everywhere can be obtained by matching method [13].

7 Notes to bibliography

Applications of the phenomenon similar to autoresonance was firstly suggested by Veksler [1,2], Andronov – Gorelik [3] and McMillan [4]. In particular, threshold for entering the autoresonance was pointed in [2]. Later, different systems were studied and entering the autoresonance was found in different cases, see for instance [5-10]. However accurate mathematical analysis of the autoresonance phenomenon does not undertake up to now.

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